Some Invariants of the Ternary Quartic.

By H. Ivah Thomsen.

In his discussion of the ternary quartic, Salmon* has used with advantage the special form $u=ax_1^4+bx_2^4+cx_3^4+6fx_2^2x_3^2+6gx_3^2x_1^2+6hx_1^2x_2^2$. For this form he gives the Hessian

$$h=\bar{a}x_1^6+\bar{b}x_2^6+\bar{c}x_3^6+\bar{a}_2x_1^4x_2^2+\bar{a}_3x_1^4x_3^2+\bar{b}_1x_2^4x_1^2+\bar{b}_3x_2^4x_3^2+\bar{c}_1x_3^4x_1^2+\bar{c}_2x_3^4x_2^2+\overline{m}x_1^2x_2^2x_3^2,$$

where
$$\begin{aligned} &\bar{a}=agh, && \bar{a}_2=a(bg+hf)-3gh^2, && \bar{a}_3=a(ch+fg)-3hg^2,\\ &\bar{b}=bhf, && \bar{b}_3=b(ch+fg)-3hf^2, && \bar{b}_1=b(af+gh)-3fh^2,\\ &\bar{c}=cfg, && \bar{c}_1=c(af+gh)-3fg^2, && \bar{c}_2=c(bg+hf)-3gf^2,\end{aligned}$$
$$\overline{m}=L-3P+18R,$$

and the covariant $S=a_sx_1^4+b_sx_2^4+c_sx_3^4+6f_sx_2^2x_3^2+6g_sx_3^2x_1^2+6h_sx_1^2x_2^2$, where

$$\begin{aligned} &a_s=6g^2h^2, && f_s=bcgh-f(bg^2+ch^2+R),\\ &b_s=6h^2f^2, && g_s=cahf-g(ch^2+af^2+R),\\ &c_s=6f^2g^2, && h_s=abfg-h(af^2+bg^2+R).\end{aligned}$$

Salmon does not give the discriminant of the quartic, though for the special form it may easily be calculated. Thus,

$$u_1=x_1(ax_1^2+3hx_2^2+3gx_3^2),\quad u_2=x_2(3hx_1^2+bx_2^2+3fx_3^2),\quad u_3=x_3(3gx_1^2+3fx_2^2+cx_3^2).$$

By inspection, we see that if a, b or $c=0$, u has one node; if $bc-9f^2$, or $ca-9g^2$ or $ab-9h^2=0$, u has two nodes; if

$$\Delta=\begin{vmatrix} a & 3h & 3g\\ 3h & b & 3f\\ 3g & 3f & c\end{vmatrix}=L-9P+54R=0,$$

u is the product of two conics and has four nodes. Hence, we infer that the discriminant

$$K=L(bc-9f^2)^2(ca-9g^2)^2(ab-9h^2)^2\Delta^4=L(81Q+L^2-9LP-729R^2)^2\Delta^4.$$

We verify this inference by forming K according to the well-known method based on the fact that a double point of u is also a double point of h.† This method gives

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* Salmon, "Treatise on the Higher Plane Curves," 3d Ed., Dublin, 1879, Articles 292–302. References to Salmon are to these articles unless otherwise specified. We use his abbreviations $L=abc$, $P=af^2+bg^2+ch^2$, $R=fgh$ and $Q=bcg^2h^2+cah^2f^2+abf^2g^2$.

† Salmon-Fiedler, "Alg. der lin. Trans.," Art. 90.

$K =$

$x_1^3x_2x_3$	$x_2^3x_3x_1$	$x_3^3x_1x_2$	x_1^5	x_2^5	x_3^5	$x_2^4x_1$	$x_3^4x_2$	$x_1^4x_3$	$x_3^4x_1$	$x_1^4x_2$	$x_2^4x_3$	$x_1^3x_2^2$	$x_2^3x_3^2$	$x_3^3x_1^2$	$x_1^3x_3^2$	$x_2^3x_1^2$	$x_3^3x_2^2$	$x_2^2x_3^2x_1$	$x_3^2x_1^2x_2$	$x_1^2x_2^2x_3$
a	$3h$	$3g$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$3h$	b	$3f$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$3g$	$3f$	c	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	a	0	0	0	0	0	0	0	0	$3h$	0	0	$3g$	0	0	0	0	0
0	0	0	0	b	0	0	0	0	0	0	0	0	$3f$	0	0	$3h$	0	0	0	0
0	0	0	0	0	c	0	0	0	0	0	0	0	0	$3g$	0	0	$3f$	0	0	0
0	0	0	0	0	0	$3h$	0	0	0	0	0	a	0	0	0	0	0	$3g$	0	0
0	0	0	0	0	0	0	$3f$	0	0	0	0	0	b	0	0	0	0	0	$3h$	0
0	0	0	0	0	0	0	0	$3g$	0	0	0	0	0	c	0	0	0	0	0	$3f$
0	0	0	0	0	0	0	0	0	$3g$	0	0	0	0	0	a	0	0	$3h$	0	0
0	0	0	0	0	0	0	0	0	0	$3h$	0	0	0	0	0	b	0	0	$3f$	0
0	0	0	0	0	0	0	0	0	0	0	$3f$	0	0	0	0	0	c	0	0	$3g$
0	0	0	0	0	0	0	0	a	0	0	0	0	0	$3g$	0	0	0	0	0	$3h$
0	0	0	0	0	0	b	0	0	0	0	0	$3h$	0	0	0	0	0	$3f$	0	0
0	0	0	0	0	0	0	c	0	0	0	0	0	$3f$	0	0	0	0	0	$3g$	0
0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	$3h$	0	0	$3g$	0
0	0	0	0	0	0	0	0	0	0	0	b	0	0	0	0	0	$3f$	0	0	$3h$
0	0	0	0	0	0	0	0	0	c	0	0	0	0	0	$3g$	0	0	$3f$	0	0
0	0	0	$3\bar{a}$	0	0	$\bar{b}_1$	0	0	$\bar{c}_1$	0	0	$2\bar{a}_2$	0	0	$2\bar{a}_3$	0	0	$\overline{m}$	0	0
0	0	0	0	$3\bar{b}$	0	0	$\bar{c}_2$	0	0	$\bar{a}_2$	0	0	$2\bar{b}_3$	0	0	$2\bar{b}_1$	0	0	$\overline{m}$	0
0	0	0	0	0	$3\bar{c}$	0	0	$\bar{a}_3$	0	0	$\bar{b}_3$	0	0	$2\bar{c}_1$	0	0	$2\bar{c}_2$	0	0	$\overline{m}$

Add to row 19 row 4 multiplied by $-3gh$ and perform the obvious similar operations. Then, we see that K is the product of L, Δ and a 15-row determinant. To reduce this determinant, add to row 13 row 1 multiplied by hf, row 8 multiplied by $-(af+gh)$, row 4 multiplied by fg, row 12 multiplied by $-(af+gh)$; multiply row 1 by b and add row 8 multiplied by $-3h$. Perform the similar operations. Then K is seen to be equal to the product of L, Δ, $(ab-9h^2)^2\,(bc-9f^2)^2\,(ca-9g^2)^2$ and a 9-row determinant. If in this determinant we add to row 7 row 1 multiplied by $-2g$ and row 4 multiplied by $-2h$, etc., we have

$$K=L\Delta^4(81Q+L^2-9LP-729R^2)^2.$$

Since s is of the same form as u, the discriminant of s is

$$K_s=L_s\Delta_s^4(b_sc_s-9f_s^2)^2\,(c_sa_s-9g_s^2)^2\,(a_sb_s-9h_s^2)^2.$$

Salmon gives

$$L_s=216R^4,$$
$$P_s=6\,[Q^2-2PRQ-4R^2Q+2P^2R^2-2PLR^2+4PR^3+6LR^3+3R^4],$$
$$R_s=Q^2-2LRQ-P^2R^2-2PR^3+L^2R^2+4LR^3-R^4.$$

From these values we find

$$\Delta_s=L_s-9P_s+54R_s=-54RM(2Q-R(L+3P)),$$

where

$$M=L-P-2R.$$

As to the remaining factors

$$b_sc_s-9f_s^2=36f^2R^2-9f_s^2=-9\,(f_s+2fR)\,(f_s-2fR)\,;$$

and

$$f_s+2fR=(bg-hf)\,(ch-fg),$$

so that

$$(f_s+2fR)\,(g_s+2gR)\,(h_s+2hR)=(Q-R(L+P-R))^2.$$

We will write

$$(f_s-2fR)\,(g_s-2gR)\,(h_s-2hR)=V,$$

and find by direct calculation

$$V=Q^2-2\,(L-P-3R)\,QR+R^2(L^2-2LP-3P^2+10LR-18PR-27R^2).$$

We have thus shown that

$$K_s=\rho R^8M^4(Q-R(L+P-R))^4\,(2Q-R(L+3P))^4\,V^2.^*$$

Referring to the invariants given by Salmon, we find by direct calculation

$$E_1-AB^2=16R^2M\,(Q-R\,(L+P-R))\,;$$

* By ρ we understand a numerical factor which it is not necessary to specify.

hence it follows that, for the special form we have used, $E_1 - AB^2$ is a factor of K_s. Dr. Coble has shown that this is true in general, since if $E_1 - AB^2 = 0$ s consists of two conics.*

We now consider the remaining factor of K_s; we call it S_2 and have

$$S_2 = \rho(2Q - R(L+3P))^4 V^2.$$

It is three conditions on a conic that it be a repeated line; hence, it is one condition on a ternary n-ic that the polar conic of some point in regard to it be a repeated line. If a cubic has this property, it is catalectic.

If the polar conic of a point y as to a quartic u is a repeated line, and y is not on the line, we may take the line as the side x_2 of the reference triangle and y as the opposite vertex, e_2. Then,

$$\begin{aligned} u &= ax_1^4 + bx_2^4 + cx_3^4 + 6gx_3^2x_1^2 + 12lx_1^2x_2x_3 + 12nx_3^2x_1x_2 \\ &\qquad + 4a_2x_1^3x_2 + 4a_3x_1^3x_3 + 4c_1x_3^3x_1 + 4c_2x_3^3x_2, \\ u_2 &= a_2x_1^3 + bx_2^3 + c_2x_3^3 + 3lx_1^2x_3 + 3nx_3^2x_1. \end{aligned}$$

If we take for x_1 and x_3 the lines joining e_2 to the Hessian points of the points in which x_2 meets u_2, we shall have $l = n = 0$, and

$$u = ax_1^4 + bx_2^4 + cx_3^4 + 6gx_3^2x_1^2 + 4a_2x_1^3x_2 + 4a_3x_1^3x_3 + 4c_1x_3^3x_1 + 4c_2x_3^3x_2.$$

For this form

$$\begin{aligned} u_1 &= ax_1^3 + c_1x_3^3 + 3a_2x_1^2x_2 + 3a_3x_1^2x_3 + 3gx_3^2x_1, \\ u_2 &= a_2x_1^3 + bx_2^3 + c_2x_3^3, \\ u_3 &= a_3x_1^3 + cx_3^3 + 3gx_1^2x_3 + 3c_1x_3^2x_1 + 3c_2x_3^2x_2, \\ -S &= a_2^2c_2^2x_1^2x_3^2 - bx_2[a_2(g^2 - a_3c_1)x_1^3 + (g(a_2c_1 - ac_2) + a_3(a_3c_2 - ca_2))x_1^2x_3 \\ &\qquad + (g(c_2a_3 - ca_2) + c_1(a_2c_1 - ac_2))x_1x_3^2 + c_2(g^2 - a_3c_1)x_3^3] \\ &\qquad + ba_2c_2x_2^2(a_3x_1^2 + 2gx_1x_3 + c_1x_3^2), \\ h &= bx_2^2[(ax_1^2 + 2a_3x_1x_3 + gx_3^2 + 2a_2x_1x_2)(gx_1^2 + 2c_1x_1x_3 + cx_3^2 + 2c_2x_2x_3) \\ &\qquad - (a_3x_1^2 + 2gx_1x_3 + c_1x_3^2)^2] + 2a_2c_2x_1^2x_3^2(a_3x_1^2 + 2gx_1x_3 + c_1x_3^2) \\ &\qquad - a_2^2x_1^4(gx_1^2 + 2c_1x_1x_3 + cx_3^2 + 2c_2x_2x_3) - c_2^2x_3^4(ax_1^2 + 2a_3x_1x_3 + gx_3^2 + 2a_2x_1x_2). \end{aligned}$$

Hence, we see that the polar cubic of every point on $x_2 = 0$ has a double point at e_2; the cuspidal cubics corresponding to e_1 and e_3; x_2 is a factor of the Steinerian of u; S has a node at e_2, the nodal tangents being $u_{12} = 0$; h has a node at e_2, the nodal tangents being $x_1x_3 = 0$.

We are now in a position to write,† in the form of a 15-row determinant, an invariant of a quartic, the vanishing of which expresses the condition that

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI, p. 357.

† Assuming that the coefficients of s have been calculated.

the polar conic of some point is a repeated line; we shall prove by using Salmon's special form that this invariant is identical with S_2.

For, if the polar conic of y is a repeated line, its line equation vanishes identically. Hence, y satisfies three quartic equations such as $u_{22}u_{33}-u_{23}^2=0$, and three such as $u_{11}u_{23}-u_{31}u_{12}=0$; also y is a node of s and satisfies the three cubic equations $s_1=0$, $s_2=0$, $s_3=0$. From each cubic equation we can get three quartics, making in all fifteen quartic equations which y must satisfy. Eliminating y dialytically from these equations, we have the required determinant. It contains six rows of degree 2 and nine rows of degree 4, and, consequently, when expanded, is of degree 48 in the coefficients of u.

For the special form we have

$$u_{22}u_{33}-u_{23}^2=ghy_1^4+bfy_2^4+cfy_3^4+(bc-3f^2)y_2^2y_3^2+(ch+fg)y_3^2y_1^2+(bg+hf)y_1^2y_2^2,$$
$$u_{11}u_{23}-u_{31}u_{12}=-2[(2gh-af)y_1^2y_2y_3-hfy_2^3y_3-fgy_3^3y_2].$$

From s we get three forms such as

$$2a_sy_1^4+g_sy_3^2y_1^2+h_sy_1^2y_2^2$$

and six such as

$$g_sy_1^2y_2y_3+f_sy_2^3y_3+2c_sy_3^3y_2.$$

Hence the 15-row determinant is the product of a 6-row determinant, Δ_6, and a 9-row determinant, Δ_9, where

$$\Delta_6=\begin{vmatrix} y_1^4 & y_2^4 & y_3^4 & y_2^2y_3^2 & y_3^2y_1^2 & y_1^2y_2^2 \\ gh & bf & cf & bc-3f^2 & ch+fg & bg+hf \\ ag & hf & cg & ch+fg & ca-3g^2 & af+gh \\ ah & bh & fg & bg+hf & af+gh & ab-3h^2 \\ 2g^2h^2 & 0 & 0 & 0 & g_s & h_s \\ 0 & 2h^2f^2 & 0 & f_s & 0 & h_s \\ 0 & 0 & 2f^2g^2 & f_s & g_s & 0 \end{vmatrix}$$

$$\Delta_9=\begin{vmatrix} y_1^2y_2y_3 & y_2^2y_3y_1 & y_3^2y_1y_2 & y_1^3y_2 & y_1^3y_3 & y_2^3y_1 & y_2^3y_3 & y_3^3y_1 & y_3^3y_2 \\ 2gh-af & 0 & 0 & 0 & 0 & 0 & -hf & 0 & -fg \\ 0 & 2hf-bg & 0 & 0 & -gh & 0 & 0 & -fg & 0 \\ 0 & 0 & 2fg-ch & -gh & 0 & -hf & 0 & 0 & 0 \\ 0 & 0 & g_s & 2g^2h^2 & 0 & h_s & 0 & 0 & 0 \\ 0 & h_s & 0 & 0 & 2g^2h^2 & 0 & 0 & g_s & 0 \\ h_s & 0 & 0 & 0 & 0 & 0 & 2h^2f^2 & 0 & f_s \\ 0 & 0 & f_s & h_s & 0 & 2h^2f^2 & 0 & 0 & 0 \\ 0 & f_s & 0 & 0 & g_s & 0 & 0 & 2f^2g^2 & 0 \\ g_s & 0 & 0 & 0 & 0 & 0 & f_s & 0 & 2f^2g^2 \end{vmatrix}$$

To expand Δ_6, we multiply column 4 by gh and add to it column 2 multiplied

by $-g^2$ and column 3 multiplied by $-h^2$, and perform the obvious similar operations. Then,

$$R^2\Delta_6 = V \begin{vmatrix} gh & bf & cf & 1 & 0 & 0 \\ ag & hf & cg & 0 & 1 & 0 \\ ah & bh & fg & 0 & 0 & 1 \\ 2g^2h^2 & 0 & 0 & 0 & hf & fg \\ 0 & 2h^2f^2 & 0 & gh & 0 & fg \\ 0 & 0 & 2f^2g^2 & gh & hf & 0 \end{vmatrix}$$

Now, multiply row 1 by gh, row 2 by hf, row 3 by fg, so that gh is a factor of column 1 and of column 4, etc. Taking out these factors, thus getting rid of the factor R^2 on the left, we have readily

$$\Delta_6 = -V \begin{vmatrix} 2(af-gh) & bg+hf & ch+fg \\ af+gh & 2(bg-hf) & ch+fg \\ af+gh & bg+hf & 2(ch-fg) \end{vmatrix} = 4V(2Q-R(L+3P)).$$

As to Δ_9, the expression $(2gh-af)(f_s+2fR)+hfg_s+fgh_s$ proves, when expanded, to be symmetric and equal to $2Q-R(L+3P)$. Hence, multiplying row 1 by f_s+2fR and adding to it row 9 multiplied by hf and row 6 multiplied by fg, etc., we have

$$\begin{aligned}(f_s+2fR)(g_s+2gR)(h_s+2hR)\Delta_9 \\ = (2Q-R(L+3P))^3\end{aligned} \begin{vmatrix} 2g^2h^2 & 0 & h_s & 0 & 0 & 0 \\ 0 & 2g^2h^2 & 0 & 0 & g_s & 0 \\ 0 & 0 & 0 & 2h^2f^2 & 0 & f_s \\ h_s & 0 & 2h^2f^2 & 0 & 0 & 0 \\ 0 & g_s & 0 & 0 & 2f^2g^2 & 0 \\ 0 & 0 & 0 & f_s & 0 & 2f^2g^2 \end{vmatrix}$$

Multiplying row 4 by $2g^2h^2$ and subtracting from it row 1 multiplied by h_s, etc., we have $\Delta_9 = V(2Q-R(L+3P))^3$. Hence,

$$\Delta_6\Delta_9 = 4(2Q-R(L+3P))^4V^2 = \rho S_2.$$

If we calculate s for a cuspidal quartic, we find that a cusp of u is also a cusp of s. Hence, $S_2=0$ if u has a cusp. However, it may happen that both $K=0$ and $S_2=0$, and u is not cuspidal; *i. e.*, the cusp and the point, the polar conic of which is a repeated line, may or may not coincide.*

There is an invariant of lower order which vanishes if u has a cusp; for, if y is a cusp, it satisfies the equations $u_1=0$, $u_2=0$ and $u_3=0$, and we form the invariant by using these equations instead of the three derived from s in

* S_2 is also a factor of the discriminant of h, occurring, probably, to the second degree.

the work given above. This invariant, which we call G, is of degree 21 in the coefficients of u. For the special form the determinant, as before, consists of two factors Δ_6 and Δ_9, where

$$\Delta_6=\begin{vmatrix} gh & bf & cf & bc-3f^2 & ch+fg & bg+hf \\ ag & hf & cg & ch+fg & ca-3g^2 & af+gh \\ ah & bh & fg & bg+hf & af+gh & ab-3h^2 \\ a & 0 & 0 & 0 & 3g & 3h \\ 0 & b & 0 & 3f & 0 & 3h \\ 0 & 0 & c & 3f & 3g & 0 \end{vmatrix}$$

and

$$\Delta_9=\begin{vmatrix} 2gh-af & 0 & 0 & 0 & 0 & 0 & -hf & 0 & -fg \\ 0 & 2hf-bg & 0 & 0 & -gh & 0 & 0 & -fg & 0 \\ 0 & 0 & 2fg-ch & -gh & 0 & -hf & 0 & 0 & 0 \\ 0 & 0 & 3g & a & 0 & 3h & 0 & 0 & 0 \\ 0 & 3h & 0 & 0 & a & 0 & 0 & 3g & 0 \\ 3h & 0 & 0 & 0 & 0 & 0 & b & 0 & 3f \\ 0 & 0 & 3f & 3h & 0 & b & 0 & 0 & 0 \\ 0 & 3f & 0 & 0 & 3g & 0 & 0 & c & 0 \\ 3g & 0 & 0 & 0 & 0 & 0 & 3f & 0 & c \end{vmatrix}$$

To expand Δ_6 we multiply row 1 by a and subtract from it row 4 multiplied by gh, row 5 multiplied by af, row 6 multiplied by af, etc. Then,

$$\Delta_6=\begin{vmatrix} a(bc-9f^2), & cah-2afg-3hg^2, & abg-2ahf-3gh^2 \\ bch-2bfg-3hf^2, & b(ca-9g^2), & abf-2bgh-3fh^2 \\ bcg-2chf-3gf^2, & caf-2cgh-3fg^2, & c(ab-9h^2) \end{vmatrix}$$

$$=Q(61L-21P+102R)+L(L^2-10LP+9P^2)$$
$$+LR(14L-46P-565R)+9R^2(5P-6R).$$

To expand Δ_9 we note that if we multiply row 1 by $bc-9f^2$ and add to it row 6 multiplied by $f(ch-3fg)$ and row 9 multiplied by $f(bg-3hf)$, the first element becomes $(bc-9f^2)(3gh-af)-(3hf-bg)(3fg-ch)$, while all other elements of this row are zero.

Hence $(bc-9f^2)(ca-9g^2)(ab-9h^2)\Delta_9$ is equal to the product of three terms such as the first element into the 6-row determinant

$$\begin{vmatrix} a & 0 & 3h & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 3g & 0 \\ 0 & 0 & 0 & b & 0 & 3f \\ 3h & 0 & b & 0 & 0 & 0 \\ 0 & 3g & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 3f & 0 & c \end{vmatrix}=-(bc-9f^2)(ca-9g^2)(ab-9h^2).$$

Hence,

$$\begin{aligned}\Delta_9=&[(bc-9f^2)(3gh-af)-(3hf-bg)(3fg-ch)]\\&[(ca-9g^2)(3hf-bg)-(3fg-ch)(3gh-af)]\\&[(ab-9h^2)(3fg-ch)-(3gh-af)(3hf-bg)]\\=&8[(LR-3Q)^2+3LR(P-3R)^2]-4[3Q(P+3R)+LR(P-3R)]X\\&+2(Q+3PR)X^2-RX^3,\end{aligned}$$

where

$$X=L-3P+36R.$$

For the quartic* u the equation

$$\begin{aligned}\phi=&(u_{22}u_{33}-u_{23}^2)\xi_1^2+(u_{33}u_{11}-u_{31}^2)\xi_2^2+(u_{11}u_{22}-u_{12}^2)\xi_3^2\\&+2(u_{31}u_{12}-u_{11}u_{23})\xi_2\xi_3+2(u_{12}u_{23}-u_{22}u_{31})\xi_3\xi_1+2(u_{23}u_{31}-u_{33}u_{12})\xi_1\xi_2=0\end{aligned}$$

is, for a given x and variable ξ, the line equation of the polar conic of x. For a given ξ and variable x it is the equation of a point quartic. It is natural to ask the relation of this quartic to the curve u and the line ξ.

The polar cubic of y is $y_1u_1+y_2u_2+y_3u_3=0$, and, if y is restricted to the line ξ, so that $(y\xi)=0$, $y_1(\xi_2u_1-\xi_1u_2)+y_3(\xi_2u_3-\xi_3u_2)=0$ represents the pencil of polar cubics corresponding to the line.

If, for a given value of the ratio $y_1:y_3$, such a curve has a double point x, this point must lie on each of the three curves

$$\frac{\xi_2u_{31}-\xi_3u_{12}}{\xi_2u_{11}-\xi_1u_{12}}=\frac{\xi_2u_{23}-\xi_3u_{22}}{\xi_2u_{12}-\xi_1u_{22}}=\frac{\xi_2u_{33}-\xi_3u_{23}}{\xi_2u_{31}-\xi_1u_{23}}.$$

Reducing these equations, we have three members of the net of quartics determined by the twelve points which are double points of polar cubics of points on the line ξ, viz.:

$$\begin{aligned}\phi_1&=\xi_1(u_{22}u_{33}-u_{23}^2)+\xi_2(u_{23}u_{31}-u_{33}u_{12})+\xi_3(u_{12}u_{23}-u_{22}u_{31}),\\\phi_2&=\xi_1(u_{23}u_{31}-u_{33}u_{12})+\xi_2(u_{33}u_{11}-u_{31}^2)+\xi_3(u_{31}u_{12}-u_{11}u_{23}),\\\phi_3&=\xi_1(u_{12}u_{23}-u_{22}u_{31})+\xi_2(u_{31}u_{12}-u_{11}u_{23})+\xi_3(u_{11}u_{22}-u_{12}^2).\end{aligned}$$

Such a set of twelve points lies on h; the curve $\eta_1\phi_1+\eta_2\phi_2+\eta_3\phi_3=0$ meets h in the two sets of points corresponding to the line ξ and the line η. If, in the equation of this curve, we let $\eta_i=\xi_i$, it becomes $\phi=0$.

Hence, we would infer that ϕ touches h at the twelve points corresponding to ξ. Such, indeed, is the case since for the line $x_1=0$, $\phi=u_{22}u_{33}-u_{23}^2$, and, writing h in the form

$$u_{11}(u_{22}u_{33}-u_{23}^2)+u_{12}(u_{23}u_{31}-u_{33}u_{12})+u_{31}(u_{12}u_{23}-u_{22}u_{31})=0,$$

it is obvious that h reduces to a square if $u_{22}u_{33}-u_{23}^2=0$.

We have already stated, for the special form, the expanded value of $u_{22}u_{33}-u_{23}^2$, the coefficient of ξ_1^2 in ϕ; Salmon gives us the covariant σ. If we

* This argument may easily be extended to the ternary n-ic.

operate on one of these forms with the other, the result is

$$24\,[f(3L+5P+2R)-8af^3+4bcgh],$$

which is, omitting the factor 24, the coefficient of ξ_1^2 in the contravariant conic given by Salmon. Hence, this conic may be defined as the envelope of lines such that the curves ϕ corresponding to them are apolar to σ.

Dr. Morley suggested that an invariant of a quartic, u, could be written in the form of a 15-row determinant, the vanishing of which would express the condition that it be possible to determine a second quartic, $\bar{u}$, such that the Clebschian of u and $\bar{u}$ vanish identically. We can write down the coefficients of the Clebschian of u and $\bar{u}$ from those of the contravariant σ of u, which Salmon gives for the general form, by writing for $2bc$, $b\bar{c}+\bar{b}c$, and for f^2, $f\bar{f}$, etc.* Equating each of the coefficients of σ to zero and eliminating $\bar{a}$, etc., we have the invariant

$$E_3=\begin{array}{c}
\begin{matrix} \bar{a} & \bar{b} & \bar{c} & \bar{f} & \bar{g} & \bar{h} & \bar{l} & \bar{m} & \bar{n} & \bar{a}_2 & \bar{a}_3 & \bar{b}_1 & \bar{b}_3 & \bar{c}_1 & \bar{c}_2 \end{matrix} \\
\left|\begin{matrix}
0 & c & b & 6f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4c_2 & 0 & -4b_3 \\
c & 0 & a & 0 & 6g & 0 & 0 & 0 & 0 & 0 & -4c_1 & 0 & 0 & -4a_3 & 0 \\
b & a & 0 & 0 & 0 & 6h & 0 & 0 & 0 & -4b_1 & 0 & -4a_2 & 0 & 0 & 0 \\
f & 0 & 0 & a & h & g & 4l & -2a_3 & -2a_2 & -2n & -2m & 0 & 0 & 0 & 0 \\
0 & g & 0 & h & b & f & -2b_3 & 4m & -2b_1 & 0 & 0 & -2n & -2l & 0 & 0 \\
0 & 0 & h & g & f & c & -2c_2 & -2c_1 & 4n & 0 & 0 & 0 & 0 & -2m & -2l \\
0 & 0 & 0 & 2l & -b_3 & -c_2 & 2f & -n & -m & 0 & 0 & c_1 & -g & b_1 & -h \\
0 & 0 & 0 & -a_3 & 2m & -c_1 & -n & 2g & -l & c_2 & -f & 0 & 0 & -h & a_2 \\
0 & 0 & 0 & -a_2 & -b_1 & 2n & -m & -l & 2h & -f & b_3 & -g & a_3 & 0 & 0 \\
0 & 0 & -b_1 & -3n & 0 & 0 & 0 & -3c_2 & -3f & 0 & 0 & -c & c_1 & b_3 & -3m \\
-c_2 & 0 & 0 & 0 & -3l & 0 & -3g & 0 & -3a_3 & c_1 & -3n & 0 & 0 & a_2 & -a \\
0 & -a_3 & 0 & 0 & 0 & -3m & -3b_1 & -3h & 0 & 0 & -b & -3l & a_2 & 0 & b_3 \\
0 & -c_1 & 0 & -3m & 0 & 0 & 0 & -3f & -3b_3 & 0 & 0 & c_2 & -3n & -b & b_1 \\
0 & 0 & -a_2 & 0 & -3n & 0 & -3c_1 & 0 & -3g & -c & c_2 & 0 & 0 & -3l & a_3 \\
-b_3 & 0 & 0 & 0 & 0 & -3l & -3h & -3a_2 & 0 & -3m & b_1 & a_3 & -a & 0 & 0
\end{matrix}\right|
\end{array}$$

For the special form we have $E_3=\Delta_6\Delta_9$, where

$$\Delta_6=\begin{vmatrix}
0 & c & b & 6f & 0 & 0 \\
c & 0 & a & 0 & 6g & 0 \\
b & a & 0 & 0 & 0 & 6h \\
f & 0 & 0 & a & h & g \\
0 & g & 0 & h & b & f \\
0 & 0 & h & g & f & c
\end{vmatrix}
\qquad
\Delta_9=\begin{vmatrix}
2f & 0 & 0 & 0 & 0 & 0 & -g & 0 & -h \\
0 & 2g & 0 & 0 & -f & 0 & 0 & -h & 0 \\
0 & 0 & 2h & -f & 0 & -g & 0 & 0 & 0 \\
0 & 0 & -3f & 0 & 0 & -c & 0 & 0 & 0 \\
-3g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a \\
0 & -3h & 0 & 0 & -b & 0 & 0 & 0 & 0 \\
0 & -3f & 0 & 0 & 0 & 0 & 0 & -b & 0 \\
0 & 0 & -3g & -c & 0 & 0 & 0 & 0 & 0 \\
-3h & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0
\end{vmatrix}$$

* This may be verified by calculation; obviously the Clebschian must reduce to a numerical multiple of σ if $u\equiv\bar{u}$.

To expand Δ_6 multiply row 1 by gh, and subtract from it row 5 multiplied by ch and row 6 multiplied by bg. We then have

$$R\Delta_6 = -\begin{vmatrix} P-6R-af^2 & b(fg+ch) & c(hf+bg) \\ a(fg+ch) & P-6R-bg^2 & c(gh+af) \\ a(hf+bg) & b(gh+af) & P-6R-ch^2 \end{vmatrix},$$

and on expansion

$$\Delta_6 = -2[L^2+2LP-3P^2+20LR+36PR-108R^2].$$

To expand Δ_9 multiply row 1 by a, and subtract from it row 5 multiplied by h and row 9 multiplied by g, when we have immediately

$$\begin{aligned} \Delta_9 &= -8L(af+3gh)(bg+3hf)(ch+3fg) \\ &= -8L(3Q+R(L+9P+27R)). \end{aligned}$$

Referring to the invariants given by Salmon, it will be found that

$$9AD_3+3A^2C_1-27E_1+A^3B+135AB^2=\rho E_3.$$

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